HESSIAN AND GRADIENT ESTIMATES FOR THREE DIMENSIONAL SPECIAL LAGRANGIAN EQUATIONS WITH LARGE PHASE

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Abstract. We derive a priori interior Hessian and gradient estimates for special Lagrangian equation of phase at least a critical value in dimension three.

1. Introduction

In this paper, we establish a priori *interior* Hessian and gradient estimates for the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{1.1}$$

with (the phase) $|\Theta| \geq \pi/2$ and $n = 3$, where λ_i are the eigenvalues of the Hessian $D^2 u$.

Equation (1.1) is from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1 + \sqrt{-1}\lambda_1\right) \cdots \left(1 + \sqrt{-1}\lambda_n\right)$ is constant Θ, that is, u satisfies equation (1.1), and it is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17]. Note that equation (1.1) with $n = 3$ and $|\Theta| = \pi/2$ or π also takes the following forms respectively

$$\sigma_2\left(D^2 u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$$

or

$$\triangle u = \det D^2 u. \tag{1.2}$$

We first state the following interior Hessian estimates.

Theorem 1.1. *Let u be a smooth solution to (1.1) with $|\Theta| \geq \pi/2$ and $n = 3$ on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2 u(0)| \leq C(3) \exp\left[C(3)\left(\cot\frac{|\Theta| - \pi/2}{3}\right)^2 \max_{B_R(0)} |Du|^7 / R^7\right],$$

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and also

$$|D^2u(0)| \le C(3)\exp\left\{C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right]\right\}.$$

The above Θ-independent Hessian estimates make use of the following Hessian estimate for the three dimensional special Lagrangian equation (1.1) with the critical phase $|\Theta| = \pi/2$.

Theorem 1.2 ([WY2])**.** *Let u be a smooth solution to (1.1) with $|\Theta| = \pi/2$ and $n = 3$ on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \le C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right].$$

In order to link the dependence of Hessian estimates in the above theorems to the potential u itself, we have the following gradient estimate in general dimensions.

Theorem 1.3. *Let u be a smooth solution to (1.1) with $|\Theta| \ge (n-2)\frac{\pi}{2}$ on $B_{3R}(0) \subset \mathbb{R}^n$. Then we have*

$$\max_{B_R(0)}|Du| \le C(n)\left[\operatorname*{osc}_{B_{3R}(0)}\frac{u}{R} + 1\right]. \tag{1.3}$$

One quick consequence of the above estimates is a Liouville type result for global solutions with quadratic growth to (1.1) with $|\Theta| = \pi/2$ and $n = 3$, namely any such a solution must be quadratic (cf. [Y1], [Y2] where other Liouville type results for convex solutions to (1.1) and Bernstein type results for global solutions to (1.1) with $|\Theta| > (n-2)\pi/2$ were obtained). Another application is the regularity (analyticity) of the C^0 viscosity solutions to (1.1) with $|\Theta| \ge \pi/2$ and $n = 3$.

In the 1950's, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère type equation including (1.1) with $n = 2$; see also Pogorelov [P1] for Hessian estimates for these equations including (1.1) with $|\Theta| > \pi/2$ and $n = 2$. In the 1970's Pogorelov [P2] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$; see generalizations of the counterexamples for σ_k equations with $k \ge 3$ in Urbas [U1]. In passing, we also mention Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equation $(k \ge 2)$ by Pogorelov [P2] and Chou-Wang [CW] respectively using the Pogorelov technique. Urbas [U2][U3], also Bao and Chen [BC] obtained (pointwise) Hessian estimates in term of certain integrals of the Hessian, for σ_k equations and special Lagrangian equation (1.1) with $n = 3$, $\Theta = \pi$ respectively.

A Hessian bound for (1.1) with $n = 2$ also follows from an earlier work by Gregori [G], where Heinz's Jacobian estimate was extended to get a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension. A gradient estimate for general dimensional

and codimensional minimal graphs with certain constraints on the gradients themselves was obtained in [W], using an integral method developed for codimension one minimal graphs. The gradient estimate of Bombieri-De Giorgi-Miranda [BDM] (see also [T1] [BG] [K]) is by now classic.

The Bernstein-Pogorelov-Korevaar technique was employed to derive Hessian estimates for (1.1) with certain constraints on the solutions in [WY1]. A slightly sharper Hessian estimate for (1.1) with $n = 2$ was obtained by elementary methods in [WY3]. The Hessian estimate for the equation $\sigma_2\left(D^2u\right) = 1$ in dimension three, or (1.1) with $|\Theta| = \pi/2$ and $n = 3$ was derived by "less" involved arguments in [WY2].

The heuristic ideas for Hessian estimates are as follows. The function $b = \ln\sqrt{1+\lambda_{\max}^2}$ is subharmonic so that b at any point is bounded by its integral over a ball around the point on the minimal surface by Michael-Simon's mean value inequality [MS]. This special choice of b is not only subharmonic, but even stronger, satisfies a Jacobi inequality. Coupled with Sobolev inequalities for functions both with and without compact support, this Jacobi inequality leads to a bound on the integral of b by the volume of the ball on the minimal surface. Taking advantage of the divergence form of the volume element of the minimal Lagrangian graph, we bound the volume in terms of the height of the special Lagrangian graph, which is the gradient of the solution to equation (1.1).

As for the gradient estimates, we adapt Trudinger's method [T2] for σ_k equations to (1.1) with the critical phase $\Theta = (n-2)\,\pi/2$. Gradient estimates for (1.1) with larger phase $\Theta > (n-2)\,\pi/2$ are straightforward consequences of the observation that the Hessians of solutions have lower bound depending on the phase Θ. In order to obtain the uniform gradient estimates independent of the phase Θ, we make use of the Lewy rotation, which links the corresponding estimates to the ones in the case of the critical phase.

Lewy rotation is also used along with a relative isoperimetric inequality to get another key ingredient in our proof of the Hessian estimates, namely a Sobolev inequality for functions without compact support, in the super critical phase case.

As one can see, our arguments for the Hessian estimates resemble the "isoperimetric" proof of the classical gradient estimate for minimal graphs. Now only some technical obstacles remain for Hessian estimates for (1.1) with large phase $|\Theta| \geq (n-2)\,\pi/2$ and $n \geq 4$. Yet further new ideas are lacking for us to handle the special Lagrangian equation (1.1) with general phases in dimension three and higher, including (1.2) corresponding to $\Theta = 0$ and $n = 3$.

Notation. $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, u_i = \partial_{ij} u$, etc., but $\lambda_1, \cdots, \lambda_n$ and $b_1 = \ln\sqrt{1+\lambda_1^2}$, $b_2 = \left(\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}\right)/2$ do not represent the partial derivatives. The eigenvalues are ordered $\lambda_1 \geq \cdots \geq \lambda_n$ and $\theta_i = \arctan\lambda_i$.

Further, h_{ijk} will denote (the second fundamental form)

$$h_{ijk}=\frac{1}{\sqrt{1+\lambda_i^2}}\frac{1}{\sqrt{1+\lambda_j^2}}\frac{1}{\sqrt{1+\lambda_k^2}}u_{ijk}.$$

when D^2u is diagonalized. Finally $C(n)$ will denote various constants depending only on dimension n.

2. Preliminaries

Taking the gradient of both sides of the special Lagrangian equation (1.1), we have

$$\sum_{i,j}^{n} g^{ij}\partial_{ij}\left(x,Du\left(x\right)\right)=0, \tag{2.1}$$

where $\left(g^{ij}\right)$ is the inverse of the induced metric $g=\left(g_{ij}\right)=I+D^2uD^2u$ on the surface $\left(x,Du\left(x\right)\right)\subset\mathbb{R}^n\times\mathbb{R}^n$. Simple geometric manipulation of (2.1) yields the divergence form of the minimal surface equation

$$\triangle_g\left(x,Du\left(x\right)\right)=0,$$

where the Laplace-Beltrami operator of the metric g is given by

$$\triangle_g=\frac{1}{\sqrt{\det g}}\sum_{i,j}^{n}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j\right).$$

Because we are using harmonic coordinates $\triangle_g x=0$, we see that $\triangle_g$ also equals the linearized operator of the special Lagrangian equation (1.1) at u,

$$\triangle_g=\sum_{i,j}^{n}g^{ij}\partial_{ij}.$$

The gradient and inner product with respect to the metric g are

$$\nabla_g v=\left(\sum_{k=1}^{n}g^{1k}v_k,\cdots,\sum_{k=1}^{n}g^{nk}v_k\right),$$

$$\left\langle\nabla_g v,\nabla_g w\right\rangle_g=\sum_{i,j=1}^{n}g^{ij}v_iw_j,\quad\text{in particular}\quad\left|\nabla_g v\right|^2=\left\langle\nabla_g v,\nabla_g v\right\rangle_g.$$

2.1. **Jacobi inequality.** We begin with some geometric calculations.

Lemma 2.1. *Let u be a smooth solution to (1.1). Suppose that the Hessian D^2u is diagonalized and the eigenvalue λ_1 is distinct from all other eigenvalues of D^2u at point p. Set $b_1=\ln\sqrt{1+\lambda_1^2}$ near p. Then we have at p*

$$\left|\nabla_g b_1\right|^2=\sum_{k=1}^{n}\lambda_1^2h_{11k}^2 \tag{2.2}$$

and

$$
\begin{aligned}
&\triangle_g b_1 = \\
&(1+\lambda_1^2)h_{111}^2 + \sum_{k>1}\left(\frac{2\lambda_1}{\lambda_1-\lambda_k} + \frac{2\lambda_1^2\lambda_k}{\lambda_1-\lambda_k}\right)h_{kk1}^2 \qquad (2.3)\\
&+\sum_{k>1}\left[1+\frac{2\lambda_1}{\lambda_1-\lambda_k} + \frac{\lambda_1^2\left(\lambda_1+\lambda_k\right)}{\lambda_1-\lambda_k}\right]h_{11k}^2 \qquad (2.4)\\
&+\sum_{k>j>1}2\lambda_1\left[\frac{1+\lambda_k^2}{\lambda_1-\lambda_k} + \frac{1+\lambda_j^2}{\lambda_1-\lambda_j} + (\lambda_j+\lambda_k)\right]h_{kj1}^2. \qquad (2.5)
\end{aligned}
$$

Proof. We first compute the derivatives of the smooth function b_1 near p. We may implicitly differentiate the characteristic equation

$$\det(D^2u - \lambda_1 I) = 0$$

near any point where λ_1 is distinct from the other eigenvalues. Then we get at p

$$
\begin{aligned}
\partial_e\lambda_1 &= \partial_e u_{11},\\
\partial_{ee}\lambda_1 &= \partial_{ee}u_{11} + \sum_{k>1}2\frac{\left(\partial_e u_{1k}\right)^2}{\lambda_1-\lambda_k},
\end{aligned}
$$

with arbitrary unit vector $e\in\mathbb{R}^n$.

Thus we have (2.2) at p

$$\left|\nabla_g b_1\right|^2 = \sum_{k=1}^n g^{kk}\left(\frac{\lambda_1}{1+\lambda_1^2}\partial_k u_{11}\right)^2 = \sum_{k=1}^n \lambda_1^2 h_{11k}^2,$$

where we used the notation $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$.

From

$$\partial_{ee}b_1 = \partial_{ee}\ln\sqrt{1+\lambda_1^2} = \frac{\lambda_1}{1+\lambda_1^2}\partial_{ee}\lambda_1 + \frac{1-\lambda_1^2}{\left(1+\lambda_1^2\right)^2}\left(\partial_e\lambda_1\right)^2,$$

we conclude that at p

$$\partial_{ee}b_1 = \frac{\lambda_1}{1+\lambda_1^2}\left[\partial_{ee}u_{11} + \sum_{k>1}2\frac{\left(\partial_e u_{1k}\right)^2}{\lambda_1-\lambda_k}\right] + \frac{1-\lambda_1^2}{\left(1+\lambda_1^2\right)^2}\left(\partial_e u_{11}\right)^2,$$

and

$$
\begin{aligned}
\triangle_g b_1 &= \sum_{\gamma=1}^n g^{\gamma\gamma}\partial_{\gamma\gamma}b_1\\
&= \sum_{\gamma=1}^n g^{\gamma\gamma}\frac{\lambda_1}{1+\lambda_1^2}\left(\partial_{\gamma\gamma}u_{11} + \sum_{k>1}2\frac{\left(u_{1k\gamma}\right)^2}{\lambda_1-\lambda_k}\right) + \sum_{\gamma=1}^n\frac{1-\lambda_1^2}{\left(1+\lambda_1^2\right)^2}g^{\gamma\gamma}u_{11\gamma}^2. \qquad (2.6)
\end{aligned}
$$

Next we substitute the fourth order derivative terms $\partial_{\gamma\gamma}u_{11}$ in the above by lower order derivative terms. Differentiating the minimal surface equation (2.1) $\sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}u_{j\alpha\beta}=0$, we obtain

$$\begin{aligned}\triangle_g u_{ij} &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}u_{ji\alpha\beta}=\sum_{\alpha,\beta=1}^{n} -\partial_i g^{\alpha\beta}u_{j\alpha\beta}=\sum_{\alpha,\beta,\gamma,\delta=1}^{n} g^{\alpha\gamma}\partial_i g_{\gamma\delta}g^{\delta\beta}u_{j\alpha\beta}\\ &=\sum_{\alpha,\beta=1}^{n} g^{\alpha\alpha}g^{\beta\beta}(\lambda_\alpha+\lambda_\beta)u_{\alpha\beta i}u_{\alpha\beta j}, \qquad (2.7)\end{aligned}$$

where we used

$$\partial_i g_{\gamma\delta}=\partial_i(\delta_{\gamma\delta}+\sum_{\varepsilon=1}^{n}u_{\gamma\varepsilon}u_{\varepsilon\delta})=u_{\gamma\delta i}(\lambda_\gamma+\lambda_\delta)$$

with diagonalized D^2u. Plugging (2.7) with $i=j=1$ in (2.6), we have at p

$$\begin{aligned}\triangle_g b_1 &= \frac{\lambda_1}{1+\lambda_1^2}\left[\sum_{\alpha,\beta=1}^{n} g^{\alpha\alpha}g^{\beta\beta}(\lambda_\alpha+\lambda_\beta)u_{\alpha\beta1}^2+\sum_{\gamma=1}^{n}\sum_{k>1}2\frac{u_{1k\gamma}^2}{\lambda_1-\lambda_k}g^{\gamma\gamma}\right]\\ &\quad+\sum_{\gamma=1}^{n}\frac{1-\lambda_1^2}{\left(1+\lambda_1^2\right)^2}g^{\gamma\gamma}u_{11\gamma}^2\\ &=\lambda_1\sum_{\alpha,\beta=1}^{n}(\lambda_\alpha+\lambda_\beta)h_{\alpha\beta1}^2+\sum_{k>1}\sum_{\gamma=1}^{n}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{\gamma k1}^2+\sum_{\gamma=1}^{n}\left(1-\lambda_1^2\right)h_{11\gamma}^2,\end{aligned}$$

where we used the notation $h_{ijk}=\sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$. Regrouping those terms $h_{\heartsuit\heartsuit1}$, $h_{11\heartsuit}$, and $h_{\heartsuit\clubsuit1}$ in the last expression, we have

$$\begin{aligned}\triangle_g b_1 &= \left(1-\lambda_1^2\right)h_{111}^2+\sum_{\alpha=1}^{n}2\lambda_1\lambda_\alpha h_{\alpha\alpha1}^2+\sum_{k>1}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{kk1}^2\\ &+\sum_{k>1}2\lambda_1(\lambda_k+\lambda_1)h_{k11}^2+\sum_{k>1}\left(1-\lambda_1^2\right)h_{11k}^2+\sum_{k>1}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{1k1}^2\\ &+\sum_{k>j>1}2\lambda_1(\lambda_j+\lambda_k)h_{jk1}^2+\sum_{\substack{j,k>1,\\ j\neq k}}\frac{2\lambda_1\left(1+\lambda_k^2\right)}{\lambda_1-\lambda_k}h_{jk1}^2.\end{aligned}$$

After simplifying the above expression, we have the second formula in Lemma 2.1. $\square$

Lemma 2.2. *Let u be a smooth solution to (1.1) with $n=3$ and $\Theta\geq\pi/2$. Suppose that the ordered eigenvalues $\lambda_1\geq\lambda_2\geq\lambda_3$ of the Hessian D^2u satisfy $\lambda_1>\lambda_2$ at point p. Set*

$$b_1=\ln\sqrt{1+\lambda_{\max}^2}=\ln\sqrt{1+\lambda_1^2}$$

Then we have at p

$$\triangle_g b_1 \geq \frac{1}{3}|\nabla_g b_1|^2. \tag{2.8}$$

Proof . We assume that the Hessian D^2u is diagonalized at point p.

Step 1. Recall $\theta_i = \arctan\lambda_i \in (-\pi/2, \pi/2)$ and $\theta_1 + \theta_2 + \theta_3 = \Theta \geq \pi/2$. It is easy to see that $\theta_1 \geq \theta_2 > 0$ and $\theta_i + \theta_j \geq 0$ for any pair. Consequently $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_i + \lambda_j \geq 0$ for any pair of distinct eigenvalues. It follows that (2.5) in the formula for $\triangle_g b_1$ is positive, then from (2.3) and (2.4) we have the inequality

$$\triangle_g b_1 \geq \lambda_1^2 \left(h_{111}^2 + \sum_{k>1} \frac{2\lambda_k}{\lambda_1 - \lambda_k} h_{kk1}^2 \right) + \lambda_1^2 \sum_{k>1} \left(1 + \frac{2\lambda_k}{\lambda_1 - \lambda_k} \right) h_{11k}^2. \tag{2.9}$$

Combining (2.9) and (2.2) gives

$$\begin{gathered} \triangle_g b_1 - \frac{1}{3}|\nabla_g b_1|^2 \geq \\ \lambda_1^2 \left(\frac{2}{3} h_{111}^2 + \sum_{k>1} \frac{2\lambda_k}{\lambda_1 - \lambda_k} h_{kk1}^2 \right) + \lambda_1^2 \sum_{k>1} \frac{2\left(\lambda_1 + 2\lambda_k\right)}{3\left(\lambda_1 - \lambda_k\right)} h_{11k}^2. \end{gathered} \tag{2.10}$$

Step 2. We show that the last term in (2.10) is nonnegative. Note that $\lambda_1 + 2\lambda_k \geq \lambda_1 + 2\lambda_3$. We only need to show that $\lambda_1 + 2\lambda_3 \geq 0$ in the case that $\lambda_3 < 0$ or equivalently $\theta_3 < 0$. From $\theta_1 + \theta_2 + \theta_3 = \Theta \geq \pi/2$, we have

$$\frac{\pi}{2} > \theta_3 + \frac{\pi}{2} = \left(\frac{\pi}{2} - \theta_1\right) + \left(\frac{\pi}{2} - \theta_2\right) + \Theta - \frac{\pi}{2} \geq 2\left(\frac{\pi}{2} - \theta_1\right).$$

It follows that

$$-\frac{1}{\lambda_3} = \tan\left(\theta_3 + \frac{\pi}{2}\right) > 2\tan\left(\frac{\pi}{2} - \theta_1\right) = \frac{2}{\lambda_1},$$

then

$$\lambda_1 + 2\lambda_3 > 0. \tag{2.11}$$

Step 3. We show that the first term in (2.10) is nonnegative by proving

$$\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} h_{331}^2 \geq 0. \tag{2.12}$$

We only need to show it for $\lambda_3 < 0$. Directly from the minimal surface equation (2.1)

$$h_{111} + h_{221} + h_{331} = 0,$$

we bound

$$h_{331}^2 = \left(h_{111} + h_{221}\right)^2 \leq \left(\frac{2}{3} h_{111}^2 + \frac{2\lambda_2}{\lambda_1 - \lambda_2} h_{221}^2 \right) \left(\frac{3}{2} + \frac{\lambda_1 - \lambda_2}{2\lambda_2} \right).$$

It follows that

$$\frac{2}{3}h_{111}^2+\frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2+\frac{2\lambda_3}{\lambda_1-\lambda_3}h_{331}^2\geq$$
$$\left(\frac{2}{3}h_{111}^2+\frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2\right)\left[1+\frac{2\lambda_3}{\lambda_1-\lambda_3}\left(\frac{3}{2}+\frac{\lambda_1-\lambda_2}{2\lambda_2}\right)\right].$$

The last factor becomes

$$1+\frac{2\lambda_3}{\lambda_1-\lambda_3}\left(\frac{3}{2}+\frac{\lambda_1-\lambda_2}{2\lambda_2}\right)=\frac{\sigma_2}{(\lambda_1-\lambda_3)\,\lambda_2}>0.$$

The above inequality is from the observation

$$\operatorname{Re}\prod_{i=1}^{3}\left(1+\sqrt{-1}\lambda_i\right)=1-\sigma_2\leq 0$$

for $3\pi/2>\theta_1+\theta_2+\theta_3=\Theta\geq\pi/2$. Therefore (2.12) holds.

We have proved the pointwise Jacobi inequality (2.8) in Lemma 2.2. $\square$

Lemma 2.3. *Let u be a smooth solution to (1.1) with $n=3$ and $\Theta\geq\pi/2$. Suppose that the ordered eigenvalues $\lambda_1\geq\lambda_2\geq\lambda_3$ of the Hessian D^2u satisfy $\lambda_2>\lambda_3$ at point p. Set*

$$b_2=\frac{1}{2}\left(\ln\sqrt{1+\lambda_1^2}+\ln\sqrt{1+\lambda_2^2}\right).$$

Then b_2 satisfies at p

$$\triangle_g b_2\geq 0. \tag{2.13}$$

Further, suppose that $\lambda_1\equiv\lambda_2$ in a neighborhood of p. Then b_2 satisfies at p

$$\triangle_g b_2\geq\frac{1}{3}\left|\nabla_g b_2\right|^2. \tag{2.14}$$

Proof. We assume that Hessian D^2u is diagonalized at point p. We may use Lemma 2.1 to obtain expressions for both $\triangle_g\ln\sqrt{1+\lambda_1^2}$ and $\triangle_g\ln\sqrt{1+\lambda_2^2}$, whenever the eigenvalues of D^2u are distinct. From (2.3), (2.4), and (2.5), we have

$$\triangle_g\ln\sqrt{1+\lambda_1^2}+\triangle_g\ln\sqrt{1+\lambda_2^2}= \tag{2.15}$$
$$(1+\lambda_1^2)h_{111}^2+\sum_{k>1}\frac{2\lambda_1(1+\lambda_1\lambda_k)}{\lambda_1-\lambda_k}h_{kk1}^2+\sum_{k>1}\left[1+\lambda_1^2+2\lambda_1\left(\frac{1+\lambda_1\lambda_k}{\lambda_1-\lambda_k}\right)\right]h_{11k}^2$$
$$+2\lambda_1\left[\frac{1+\lambda_3^2}{\lambda_1-\lambda_3}+\frac{1+\lambda_2^2}{\lambda_1-\lambda_2}+(\lambda_3+\lambda_2)\right]h_{321}^2$$
$$+(1+\lambda_2^2)h_{222}^2+\sum_{k\neq 2}\frac{2\lambda_2(1+\lambda_2\lambda_k)}{\lambda_2-\lambda_k}h_{kk2}^2+\sum_{k\neq 2}\left[1+\lambda_2^2+2\lambda_2\left(\frac{1+\lambda_2\lambda_k}{\lambda_2-\lambda_k}\right)\right]h_{22k}^2$$
$$+2\lambda_2\left[\frac{1+\lambda_3^2}{\lambda_2-\lambda_3}+\frac{1+\lambda_1^2}{\lambda_2-\lambda_1}+(\lambda_3+\lambda_1)\right]h_{321}^2.$$

The function b_2 is symmetric in λ_1 and λ_2, thus b_2 is smooth even when $\lambda_1 = \lambda_2$, provided that $\lambda_2 > \lambda_3$. We simplify (2.15) to the following, which holds by continuity wherever $\lambda_1 \geq \lambda_2 > \lambda_3$.

$$2 \triangle_g b_2 =$$

$$(1+\lambda_1^2)h_{111}^2 + (3+\lambda_2^2+2\lambda_1\lambda_2)h_{221}^2 + \left(\frac{2\lambda_1}{\lambda_1-\lambda_3} + \frac{2\lambda_1^2\lambda_3}{\lambda_1-\lambda_3}\right)h_{331}^2 \tag{2.16}$$

$$+ (3+\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + (1+\lambda_2^2)h_{222}^2 + \left(\frac{2\lambda_2}{\lambda_2-\lambda_3} + \frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\right)h_{332}^2 \tag{2.17}$$

$$+ \left[\frac{3\lambda_1-\lambda_3+\lambda_1^2(\lambda_1+\lambda_3)}{\lambda_1-\lambda_3}\right]h_{113}^2 + \left[\frac{3\lambda_2-\lambda_3+\lambda_2^2(\lambda_2+\lambda_3)}{\lambda_2-\lambda_3}\right]h_{223}^2 \tag{2.18}$$

$$+ 2\left[1+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 + \frac{\lambda_1\left(1+\lambda_3^2\right)}{\lambda_1-\lambda_3} + \frac{\lambda_2\left(1+\lambda_3^2\right)}{\lambda_2-\lambda_3}\right]h_{123}^2. \tag{2.19}$$

Using the relations $\lambda_1 \geq \lambda_2 > 0$, $\lambda_i + \lambda_j > 0$, and $\sigma_2 \geq 1$ derived in the proof of Lemma 2.2, we see that (2.19) and (2.18) are nonnegative. We only need to justify the nonnegativity of (2.16) and (2.17) for $\lambda_3 < 0$. From the minimal surface equation (2.1), we know

$$h_{332}^2 = (h_{112}+h_{222})^2 \leq \left[(\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2 h_{222}^2\right]\left(\frac{1}{\lambda_1^2+2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right).$$

It follows that

$$\begin{aligned}(2.17) &\geq (\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2h_{222}^2 + \frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}h_{332}^2 \\ &\geq \left[(\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2h_{222}^2\right]\left[1 + \frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\left(\frac{1}{\lambda_1^2+2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right)\right].\end{aligned}$$

The last term becomes

$$\begin{aligned}&\frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\left(\frac{\lambda_2-\lambda_3}{2\lambda_2^2\lambda_3} + \frac{1}{\lambda_1^2+2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right) \\ &= \frac{\lambda_2}{\lambda_2-\lambda_3}\left[\frac{\sigma_2}{\lambda_1\lambda_2} - \frac{\lambda_3}{(\lambda_1+2\lambda_2)}\right] \geq 0.\end{aligned}$$

Thus (2.17) is nonnegative. Similarly (2.16) is nonnegative. We have proved (2.13).

Next we prove (2.14), still assuming D^2u is diagonalized at point p. Plugging in $\lambda_1 = \lambda_2$ into (2.16), (2.17), and (2.18), we get

$$2 \triangle_g b_2 \geq$$

$$\begin{aligned}&\lambda_1^2\left(h_{111}^2+3h_{221}^2+\frac{2\lambda_3}{\lambda_1-\lambda_3}h_{331}^2\right)\\&+\ \lambda_1^2\left(3h_{112}^2+h_{222}^2+\frac{2\lambda_3}{\lambda_1-\lambda_3}h_{332}^2\right)\\&+\lambda_1^2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)\left(h_{113}^2+h_{223}^2\right).\end{aligned}$$

Differentiating the eigenvector equations in the neighborhood where $\lambda_1\equiv\lambda_2$

$$\left(D^2u\right)U=\frac{\lambda_1+\lambda_2}{2}U,\ \left(D^2u\right)V=\frac{\lambda_1+\lambda_2}{2}V,\ \text{and}\ \left(D^2u\right)W=\lambda_3W,$$

we see that $u_{11e}=u_{22e}$ for any $e\in\mathbb{R}^3$ at point p. Using the minimal surface equation (2.1), we then have

$$h_{11k}=h_{22k}=-\frac{1}{2}h_{33k}$$

at point p. Thus

$$\triangle_g b_2\geq\lambda_1^2\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{111}^2+2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{112}^2+\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)h_{113}^2\right].$$

The gradient $|\nabla_g b_2|^2$ has the expression at p

$$|\nabla_g b_2|^2=\sum_{k=1}^3 g^{kk}\left(\frac{1}{2}\frac{\lambda_1}{1+\lambda_1^2}\partial_k u_{11}+\frac{1}{2}\frac{\lambda_2}{1+\lambda_2^2}\partial_k u_{22}\right)^2=\sum_{k=1}^3\lambda_1^2h_{11k}^2.$$

Thus at p

$$\begin{gathered}\triangle_g b_2-\frac{1}{3}|\nabla_g b_2|^2\geq\\ \lambda_1^2\left\{\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)-\frac{1}{3}\right]h_{111}^2+\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right)-\frac{1}{3}\right]h_{112}^2+\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}-\frac{1}{3}\right)h_{113}^2\right\}\\ \geq 0,\end{gathered}$$

where we again used $\lambda_1+2\lambda_3>0$ from (2.11). We have proved (2.14) of Lemma 2.3. □

The following is the first main result of this section. This Jacobi inequality is crucial in our proofs of Theorems 1.1 and 1.2.

Proposition 2.1. *Let u be a smooth solution to the special Lagrangian equation (1.1) with $n=3$ and $\Theta\geq\pi/2$ on B_R. Set*

$$b=\max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K\right\},$$

with $K=1+\ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)}$. Then b satisfies the integral Jacobi inequality

$$\int_{B_R}-\langle\nabla_g\varphi,\nabla_g b\rangle_g\,dv_g\geq\frac{1}{3}\int_{B_R}\varphi\,|\nabla_g b|^2\,dv_g \tag{2.20}$$

for all non-negative $\varphi \in C_0^\infty(B_R)$.

Proof. If $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is smooth everywhere, then the pointwise Jacobi inequality (2.8) in Lemma 2.2 already implies the integral Jacobi (2.20). It is known that $\lambda_{\max}$ is always a Lipschitz function of the entries of the Hessian D^2u. Now u is smooth in x, so $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is Lipschitz in terms of x. If b_1 (or equivalently $\lambda_{\max}$) is not smooth, then the first two largest eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ coincide, and $b_1(x) = b_2(x)$, where $b_2(x)$ is the average $b_2 = \left(\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}\right)/2$. We prove the integral Jacobi inequality (2.20) for a possibly singular $b_1(x)$ in two cases. Set

$$S = \{x \mid \lambda_1(x) = \lambda_2(x)\}.$$

Case 1. S has measure zero. For small $\tau > 0$, let

$$\begin{aligned}
\Omega &= B_R \backslash \{x \mid b_1(x) \le K\} = B_R \backslash \{x \mid b(x) = K\} \\
\Omega_1(\tau) &= \{x \mid b(x) = b_1(x) > b_2(x) + \tau\} \cap \Omega \\
\Omega_2(\tau) &= \{x \mid b_2(x) \le b(x) = b_1(x) < b_2(x) + \tau\} \cap \Omega.
\end{aligned}$$

Now $b(x) = b_1(x)$ is smooth in $\overline{\Omega_1(\tau)}$. We claim that $b_2(x)$ is smooth in $\overline{\Omega_2(\tau)}$. We know $b_2(x)$ is smooth wherever $\lambda_2(x) > \lambda_3(x)$. If (the Lipschitz) $b_2(x)$ is not smooth at $x_* \in \overline{\Omega_2(\tau)}$, then

$$\begin{aligned}
\ln\sqrt{1+\lambda_3^2} = \ln\sqrt{1+\lambda_2^2} &\ge \ln\sqrt{1+\lambda_1^2} - 2\tau \\
&\ge \ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)} + 1 - 2\tau,
\end{aligned}$$

by the choice of K. For small enough τ, we have $\lambda_2 = \lambda_3 > \tan\left(\frac{\Theta}{3}\right)$ and a contradiction

$$(\theta_1 + \theta_2 + \theta_3)(x_*) > \Theta.$$

Note that

$$\int_{B_R} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g = \int_\Omega -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g$$

$$= \lim_{\tau \to 0^+}\left[\int_{\Omega_1(\tau)} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g + \int_{\Omega_2(\tau)} -\langle \nabla_g \varphi, \nabla_g (b_2 + \tau)\rangle_g \, dv_g\right].$$

By the smoothness of b in $\Omega_1(\tau)$ and b_2 in $\Omega_2(\tau)$, and also inequalities (2.8) and (2.13), we have

$$\begin{aligned}
&\int_{\Omega_1(\tau)} -\langle \nabla_g\varphi, \nabla_g b\rangle_g \, dv_g + \int_{\Omega_2(\tau)} -\langle \nabla_g\varphi, \nabla_g (b_2+\tau)\rangle_g \, dv_g \\
&= \int_{\partial\Omega_1(\tau)} -\varphi \, \partial_{\nu_g^1} b \, dA_g + \int_{\Omega_1(\tau)} \varphi \triangle_g b_1 dv_g \\
&+ \int_{\partial\Omega_2(\tau)} -\varphi \partial_{\nu_g^2} (b_2+\tau) \, dA_g + \int_{\Omega_2(\tau)} \varphi \triangle_g (b_2+\tau) \, dv_g \\
&\geq \int_{\partial\Omega_1(\tau)} -\varphi \, \partial_{\nu_g^1} b \, dA_g + \int_{\partial\Omega_2(\tau)} -\varphi \partial_{\nu_g^2} (b_2+\tau) \, dA_g + \frac{1}{3}\int_{\Omega_1(\tau)} \varphi \left|\nabla_g b_1\right|^2 dv_g,
\end{aligned}$$

where ν_g^1 and ν_g^2 are the outward co-normals of $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ with respect to the metric g.

Observe that if b_1 is not smooth on any part of $\partial\Omega\backslash\partial B_R$, which is the K-level set of b_1, then on this portion $\partial\Omega\backslash\partial B_R$ is also the K-level set of b_2, which is smooth near this portion. Applying Sard's theorem, we can perturb K so that $\partial\Omega$ is piecewise C^1. Applying Sard's theorem again, we find a subsequence of positive τ going to 0, so that the boundaries $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ are piecewise C^1.

Then, we show the above boundary integrals are non-negative. The boundary integral portion along $\partial\Omega$ is easily seen non-negative, because either $\varphi = 0$, or $-\partial_{\nu_g^1} b \geq 0,\ -\partial_{\nu_g^2}(b_2+\tau) \geq 0$ there. The boundary integral portion in the interior of Ω is also non-negative, because there we have

$$\begin{gathered}
b = b_2 + \tau \quad (\text{and } b \geq b_2 + \tau \ \text{ in } \Omega_1(\tau)) \\
-\partial_{\nu_g^1} b - \partial_{\nu_g^2}(b_2+\tau) = \partial_{\nu_g^2} b - \partial_{\nu_g^2}(b_2+\tau) \geq 0.
\end{gathered}$$

Taking the limit along the (Sard) sequence of τ going to 0, we obtain $\Omega_1(\tau) \to \Omega$ up to a set of measure zero, and

$$\begin{aligned}
&\int_{B_R} -\langle \nabla_g\varphi, \nabla_g b\rangle_g \, dv_g \\
&= \int_{\Omega} -\langle \nabla_g\varphi, \nabla_g b\rangle_g \, dv_g \geq \frac{1}{3}\int_{\Omega} \left|\nabla_g b\right|^2 dv_g \\
&= \frac{1}{3}\int_{B_R} \left|\nabla_g b\right|^2 dv_g.
\end{aligned}$$

Case 2. S has positive measure. The discriminant

$$\mathcal{D} = (\lambda_1 - \lambda_2)^2 (\lambda_2 - \lambda_3)^2 (\lambda_3 - \lambda_1)^2$$

is an analytic function in B_R, because the smooth u is actually analytic (cf. [M, p. 203]). So $\mathcal{D}$ must vanish identically. Then we have either $\lambda_1(x) = \lambda_2(x)$ or $\lambda_2(x) = \lambda_3(x)$ at any point $x \in B_R$. In turn, we know that $\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan\frac{\Theta}{3}$ and $b = K > b_1(x)$ at every "boundary" point of S inside B_R, $x \in \partial S \cap \mathring{B}_R$. If the "boundary" set ∂S has positive measure, then

$\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan\frac{\Theta}{3}$ everywhere by the analyticity of u, and (2.20) is trivially true. In the case that ∂S has zero measure, $b = b_1 > K$ is smooth up to the boundary of every component of $\{x|\ b(x) > K\}$. By the pointwise Jacobi inequality (2.14), the integral inequality (2.20) is also valid in case 2. □

2.2. **Lewy rotation.** The next is the second main result of this section. Our proofs of Theorems 1.1 and 1.3 rely on this new representation of the original special Lagrangian graph.

Proposition 2.2. *Let u be a smooth solution to (1.1) with $\Theta = (n-2)\pi/2 + \delta$ on $B_R(0) \subset \mathbb{R}^n$. Then the special Lagrangian surface $\mathfrak{M} = (x, Du(x))$ can be represented as a gradient graph $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ of the new potential $\bar{u}$ satisfying (1.1) with phase $\Theta = (n-2)\pi/2$ in a domain containing a ball of radius*

$$\bar{R} \geq \frac{R}{2\cos(\delta/n)}.$$

Proof. To obtain the new representation, we use a Lewy rotation (cf. [Y1], [Y2, p. 1356]). Take a $U(n)$ rotation of $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n : \bar{z} = e^{-\sqrt{-1}\delta/n} z$ with $z = x + \sqrt{-1}y$ and $\bar{z} = \bar{x} + \sqrt{-1}\bar{y}$. Because $U(n)$ rotation preserves the length and complex structure, $\mathfrak{M}$ is still a special Lagrangian submanifold with the parametrization

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + Du(x)\sin\frac{\delta}{n} \\ D\bar{u} = -x\sin\frac{\delta}{n} + Du(x)\cos\frac{\delta}{n} \end{cases}.$$

In order to show that this parameterization is that of a gradient graph over $\bar{x}$, we must first show that $\bar{x}(x)$ is a diffeomorphism onto its image. This is accomplished by showing that

$$\left|\bar{x}(x^\alpha) - \bar{x}(x^\beta)\right| \geq \frac{1}{2\cos\delta/n}\left|x^\alpha - x^\beta\right| \tag{2.21}$$

for any x^α, x^β. We assume by translation that $x^\beta = 0$ and $Du\left(x^\beta\right) = 0$. Now $0 < \delta < \pi$, and $\theta_i > \delta - \frac{\pi}{2}$, so $u + \frac{1}{2}\cot\delta x^2$ is convex, and we have

$$\begin{aligned}
\left|\bar{x}(x^\alpha) - \bar{x}\left(x^\beta\right)\right|^2 &= \left|\bar{x}(x^\alpha)\right|^2 = \left|x^\alpha\cos\frac{\delta}{n} + Du(x^\alpha)\sin\frac{\delta}{n}\right|^2 \\
&= \left|x^\alpha\left(\cos\frac{\delta}{n} - \cot\delta\sin\frac{\delta}{n}\right) + \left[Du(x^\alpha) + x^\alpha\cot\delta\right]\sin\frac{\delta}{n}\right|^2 \\
&= |x^\alpha|^2\left[\frac{\sin\frac{(n-1)\delta}{n}}{\sin\delta}\right]^2 + \left|Du(x^\alpha) + x^\alpha\cot\delta\right|^2\sin^2\frac{\delta}{n} \\
&\quad + 2\frac{\sin\frac{(n-1)\delta}{n}\sin\frac{\delta}{n}}{\sin\delta}\left\langle x^\alpha, Du(x^\alpha) + x\cot\delta\right\rangle \\
&\geq |x^\alpha|^2\left(\frac{1}{2\cos\frac{\delta}{n}}\right)^2.
\end{aligned}$$

It follows that $\mathfrak{M}$ is a special Lagrangian graph over $\bar{x}$. The Lagrangian graph is the gradient graph of a potential function $\bar{u}$ (cf. [HL, Lemma 2.2]), that is, $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$. The eigenvalues $\bar{\lambda}_i$ of the Hessian $D^2\bar{u}$ are determined by

$$\bar{\theta}_i = \arctan\bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in \left(-\frac{\pi}{2} + \frac{(n-1)\,\delta}{n}, \frac{\pi}{2} - \frac{\delta}{n}\right). \tag{2.22}$$

Then

$$\sum_{i=1}^{n} \bar{\theta}_i = \frac{(n-2)\,\pi}{2n},$$

that is, $\bar{u}$ satisfies the special Lagrangian equation (1.1) of phase $(n-2)\,\pi/2$. The lower bound on $\bar{R}$ follows immediately from (2.21). □

2.3. **Relative isoperimetric inequality.** We end with the last main result of this section, Proposition 2.3. This relative isoperimetric inequality is needed in the proof of Theorem 1.2 to prove a key ingredient, namely a Sobolev inequality for functions without compact support. Proposition 2.3 is proved from the following classical relative isoperimetric inequality for balls.

Lemma 2.4. *Let A and A^c are disjoint measurable sets such that $A \cup A^c = B_1(0) \subset \mathbb{R}^n$. Then*

$$\min\{|A|, |A^c|\} \leq C(n)\,|\partial A \cap \partial A^c|^{n/n-1}. \tag{2.23}$$

Proof. See for example [LY, Theorem 5.3.2.]. □

Proposition 2.3. *Let $\Omega_1 \subset \Omega_2 \subset B_\rho \subset \mathbb{R}^n$. Suppose that $dist(\Omega_1, \partial\Omega_2) \geq 2$, also A and A^c are disjoint measurable sets such that $A \cup A^c = \Omega_2$. Then*

$$\min\{|A \cap \Omega_1|, |A^c \cap \Omega_1|\} \leq C(n)\rho^n\,|\partial A \cap \partial A^c|^{n/n-1}.$$

Proof. Define a continuous function on Ω_1

$$\chi(x) = \frac{|A \cap B_1(x)|}{|B_1(x)|}.$$

First, suppose that $\chi(x^*) = 1/2$ for some $x^* \in \Omega_1$. From the relative isoperimetric inequality for balls (2.23)

$$\frac{|B_1(x^*)|}{2} \leq C(n)\,|\partial A \cap \partial A^c \cap B_1(x^*)|^{n/n-1} \leq C(n)\,|\partial A \cap \partial A^c|^{n/n-1}.$$

Now

$$\min\{|A \cap \Omega_1|, |A^c \cap \Omega_1|\} \leq |\Omega_1| < |B_\rho| = \frac{|B_1(x^*)|}{2}2\rho^n \leq C(n)\rho^n\,|\partial A \cap \partial A^c|^{n/n-1},$$

and the conclusion of this proposition follows.

On the other hand, suppose that for all $x \in \Omega_1$, $\chi(x) \neq 1/2$. Then either $\chi(x) < 1/2$ on Ω_1, or $\chi(x) > 1/2$ on Ω_1. Without loss of generality we assume that $\chi(x) < 1/2$ on Ω_1. Cover Ω_2 by $C(n)\rho^n$ balls of unit radius, $B_1(x_i)$. Consider the subcover which covers Ω_1; each ball in this subcover

is completely contained inside Ω_2. Thus we may apply (2.23) to each ball in this subcover and obtain

$$|A\cap B_1(x_i)| = \min\left\{|A\cap B_1(x_i)|, |A^c\cap B_1(x_i)|\right\} \le C(n)\left|\partial A\cap \partial A^c\right|^{n/n-1}.$$

Summing this inequality over the subcover, we get

$$|A\cap\Omega_1| \le \sum_{i=1}^{C(n)\rho^n} |A\cap B_1(x_i)| \le C(n)\rho^n C(n)\left|\partial A\cap\partial A^c\right|^{n/n-1}.$$

Again, the conclusion of this proposition follows. □

Remark. Considering dumbbell type regions, we see that the order of dependence on ρ is sharp in Proposition 2.3.

3. Proof Of Theorem 1.2

For completeness, we reproduce the proof of Theorem 1.2 here. We assume that $R=4$ and u is a solution on $B_4\subset\mathbb{R}^3$ for simplicity of notation. By scaling $u\left(\frac{R}{4}x\right)/\left(\frac{R}{4}\right)^2$, we still get the estimate in Theorem 1.2. By symmetry, we assume without loss of generality that $\Theta=\pi/2$.

Step 1. By the integral Jacobi inequality (2.20) in Proposition 2.1, b is subharmonic in the integral sense, then b^3 is also subharmonic in the integral sense on the minimal surface $\mathfrak{M}=(x,Du)$:

$$\begin{aligned}\int -\left\langle\nabla_g\varphi,\nabla_g b^3\right\rangle_g dv_g &= \int -\left\langle\nabla_g\left(3b^2\varphi\right)-6b\varphi\nabla_g b,\nabla_g b\right\rangle_g dv_g\\ &\ge \int\left(\varphi b^2\left|\nabla_g b\right|^2+6b\varphi\left|\nabla_g b\right|^2\right)dv_g \ge 0 \end{aligned}\tag{3.1}$$

for all non-negative $\varphi\in C_0^\infty$, approximating b by smooth functions if necessary.

Applying Michael-Simon's mean value inequality [MS, Theorem 3.4] to the Lipschitz subharmonic function b^3, we obtain

$$b(0)\le C(3)\left(\int_{\mathfrak{B}_1\cap\mathfrak{M}} b^3 dv_g\right)^{1/3}\le C(3)\left(\int_{B_1} b^3 dv_g\right)^{1/3},$$

where $\mathfrak{B}_r$ is the ball with radius r and center $(0,Du(0))$ in $\mathbb{R}^3\times\mathbb{R}^3$, and B_r is the ball with radius r and center 0 in $\mathbb{R}^3$. Choose a cut-off function $\varphi\in C_0^\infty(B_2)$ such that $\varphi\ge 0$, $\varphi=1$ on B_1, and $|D\varphi|\le 1.1$, we then have

$$\left(\int_{B_1} b^3 dv_g\right)^{1/3}\le\left(\int_{B_2}\varphi^6 b^3 dv_g\right)^{1/3}=\left(\int_{B_2}\left(\varphi b^{1/2}\right)^6 dv_g\right)^{1/3}.$$

Applying the Sobolev inequality on the minimal surface $\mathfrak{M}$ [MS, Theorem 2.1] or [A, Theorem 7.3] to $\varphi b^{1/2}$, which we may assume to be C^1 by approximation, we obtain

$$\left(\int_{B_2}\left(\varphi b^{1/2}\right)^6 dv_g\right)^{1/3}\le C(3)\int_{B_2}\left|\nabla_g\left(\varphi b^{1/2}\right)\right|^2 dv_g.$$

Splitting the integrand as follows

$$\begin{aligned}\left|\nabla_g\left(\varphi b^{1/2}\right)\right|^2 &= \left|\frac{1}{2b^{1/2}}\varphi\nabla_g b + b^{1/2}\nabla_g\varphi\right|^2 \leq \frac{1}{2b}\varphi^2\left|\nabla_g b\right|^2 + 2b\left|\nabla_g\varphi\right|^2 \\ &\leq \frac{1}{2}\varphi^2\left|\nabla_g b\right|^2 + 2b\left|\nabla_g\varphi\right|^2,\end{aligned}$$

where we used $b \geq 1$, we get

$$\begin{aligned}b(0) &\leq C(3)\int_{B_2}\left|\nabla_g\left(\varphi b^{1/2}\right)\right|^2 dv_g \\ &\leq C(3)\left(\int_{B_2}\varphi^2\left|\nabla_g b\right|^2 dv_g + \int_{B_2} b\left|\nabla_g\varphi\right|^2 dv_g\right) \\ &\leq \underbrace{C(3)\left\|Du\right\|_{L^\infty(B_2)}}_{\text{Step 2}} + C(3)\underbrace{\left[\left\|Du\right\|^2_{L^\infty(B_3)} + \left\|Du\right\|^3_{L^\infty(B_4)}\right]}_{\text{step 3}}.\end{aligned}$$

Step 2. By (2.20) in Proposition 2.1, b satisfies the Jacobi inequality in the integral sense:

$$3\bigtriangleup_g b \geq \left|\nabla_g b\right|^2.$$

Multiplying both sides by the above non-negative cut-off function $\varphi \in C_0^\infty(B_2)$, then integrating, we obtain

$$\begin{aligned}\int_{B_2}\varphi^2\left|\nabla_g b\right|^2 dv_g &\leq 3\int_{B_2}\varphi^2\bigtriangleup_g b dv_g \\ &= -3\int_{B_2}\left\langle 2\varphi\nabla_g\varphi, \nabla_g b\right\rangle dv_g \\ &\leq \frac{1}{2}\int_{B_2}\varphi^2\left|\nabla_g b\right|^2 dv_g + 18\int_{B_2}\left|\nabla_g\varphi\right|^2 dv_g.\end{aligned}$$

It follows that

$$\int_{B_2}\varphi^2\left|\nabla_g b\right|^2 dv_g \leq 36\int_{B_2}\left|\nabla_g\varphi\right|^2 dv_g. \tag{3.2}$$

Observe the ("conformality") identity:

$$\left(\frac{1}{1+\lambda_1^2}, \frac{1}{1+\lambda_2^2}, \frac{1}{1+\lambda_3^2}\right)V = \left(\sigma_1-\lambda_1,\ \ \sigma_1-\lambda_2,\ \ \sigma_1-\lambda_3\right)$$

where we used the identity $V = \prod_{i=1}^{3}\sqrt{\left(1+\lambda_i^2\right)} = \sigma_1 - \sigma_3$ with $\sigma_2 = 1$. We then have

$$\begin{aligned}\left|\nabla_g\varphi\right|^2 dv_g &= \sum_{i=1}^{3}\frac{\left(D_i\varphi\right)^2}{1+\lambda_i^2}V dx = \sum_{i=1}^{3}\left(D_i\varphi\right)^2\left(\sigma_1-\lambda_i\right)dx \\ &\leq 2.42\bigtriangleup u\ dx.\end{aligned} \tag{3.3}$$

Thus

$$\int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g \le C(3) \int_{B_2} \triangle u \; dx$$
$$\le C(3) \|Du\|_{L^\infty(B_2)}.$$

Step 3. By (3.3), we get

$$\int_{B_2} b |\nabla_g \varphi|^2 dv_g \le C(3) \int_{B_2} b \triangle u \; dx.$$

Choose another cut-off function $\psi \in C_0^\infty(B_3)$ such that $\psi \ge 0$, $\psi = 1$ on B_2, and $|D\psi| \le 1.1$. We have

$$\begin{aligned}\int_{B_2} b \triangle u dx \le \int_{B_3} \psi b \triangle u dx &= \int_{B_3} -\langle bD\psi + \psi Db, Du\rangle \, dx \\ &\le \|Du\|_{L^\infty(B_3)} \int_{B_3} (b |D\psi| + \psi |Db|) \, dx \\ &\le C(3) \|Du\|_{L^\infty(B_3)} \int_{B_3} (b + |Db|) \, dx.\end{aligned}$$

Now

$$b = \max\left\{\ln \sqrt{1+\lambda_{\max}^2},\; K\right\} \le \lambda_{\max} + K < \lambda_1 + \lambda_2 + \lambda_3 + K = \triangle u + K,$$

where $\lambda_2 + \lambda_3 > 0$ follows from $\arctan \lambda_2 + \arctan \lambda_3 = \frac{\pi}{2} - \arctan \lambda_1 > 0$. Hence

$$\int_{B_3} b dx \le C(3)(1 + \|Du\|_{L^\infty(B_3)}).$$

And we have left to estimate $\int_{B_3} |Db| \, dx$:

$$\begin{aligned}\int_{B_3} |Db| \, dx &\le \int_{B_3} \sqrt{\sum_{i=1}^3 \frac{(b_i)^2}{(1+\lambda_i^2)} \left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)\left(1+\lambda_3^2\right)} \; dx \\ &= \int_{B_3} |\nabla_g b| \, V dx \\ &\le \left(\int_{B_3} |\nabla_g b|^2 \, V dx\right)^{1/2} \left(\int_{B_3} V dx\right)^{1/2}.\end{aligned}$$

Repeating the "Jacobi" argument from Step 2, we see

$$\int_{B_3} |\nabla_g b|^2 \, V dx \le C(3) \|Du\|_{L^\infty(B_4)}.$$

Then by the Sobolev inequality on the minimal surface $\mathfrak{M}$, we have

$$\int_{B_3} V dx = \int_{B_3} dv_g \le \int_{B_4} \phi^6 dv_g \le C(3) \left(\int_{B_4} |\nabla_g \phi|^2 \, dv_g\right)^3,$$

where the non-negative cut-off function $\phi \in C_0^\infty(B_4)$ satisfies $\phi = 1$ on B_3, and $|D\phi| \leq 1.1$. Applying the conformality equality (3.3) again, we obtain

$$\int_{B_4} |\nabla_g \phi|^2 \, dv_g \leq C(3) \int_{B_4} \triangle u \; dx \leq C(3) \|Du\|_{L^\infty(B_4)}.$$

Thus we get

$$\int_{B_3} V dx \leq C(3) \|Du\|^3_{L^\infty(B_4)}$$

and

$$\int_{B_3} |Db| \, dx \leq C(3) \|Du\|^2_{L^\infty(B_4)}.$$

In turn, we obtain

$$\int_{B_2} b |\nabla_g \varphi|^2 \, dv_g \leq C(3) \left[K \|Du\|_{L^\infty(B_3)} + \|Du\|^2_{L^\infty(B_3)} + \|Du\|^3_{L^\infty(B_4)} \right].$$

Finally collecting all the estimates in the above three steps, we arrive at

$$\begin{aligned} \lambda_{\max}(0) &\leq \exp\left[C(3) \left(\|Du\|_{L^\infty(B_4)} + \|Du\|^2_{L^\infty(B_4)} + \|Du\|^3_{L^\infty(B_4)} \right) \right] \\ &\leq C(3) \exp\left[C(3) \|Du\|^3_{L^\infty(B_4)} \right]. \end{aligned} \tag{3.4}$$

This completes the proof of Theorem 1.2.

4. Proof Of Theorem 1.1

As in the proof of Theorem 1.2, we assume that $R = 8$ and u is a solution on $B_8 \subset \mathbb{R}^3$ for simplicity of notation. By scaling $v(x) = u\left(\frac{R}{8}x\right) / \left(\frac{R}{8}\right)^2$, we still get the estimate in Theorem 1.1. We consider the cases when $\Theta = \pi/2 + \delta$ for $\delta \in (0, \pi)$. The cases $\Theta < -\pi/2$ follow by symmetry.

Step 1. As preparation for the proof of Theorem 1.2, we take the phase $\pi/2$ representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in Proposition 2.2 for the original special Lagrangian graph $\mathfrak{M} = (x, Du(x))$ with $x \in B_8$. The "critical" representation is

$$\left\{ \begin{aligned} \bar{x} &= x \cos\tfrac{\delta}{3} + Du(x) \sin\tfrac{\delta}{3} \\ D\bar{u} &= -x \sin\tfrac{\delta}{3} + Du(x) \cos\tfrac{\delta}{3} \end{aligned} \right. . \tag{4.1}$$

Define

$$\bar{\Omega}_r = \bar{x}(B_r(0)).$$

Then we have from (2.21)

$$\operatorname{dist}(\bar{\Omega}_1, \partial\bar{\Omega}_5) \geq \frac{4}{2\cos\delta/3} > 2. \tag{4.2}$$

We see from (4.1) that $|\bar{x}| \leq \rho$ for $\bar{x} \in \bar{\Omega}_8$ with

$$\rho = 8 \cos\frac{\delta}{3} + \|Du\|_{L^\infty(B_8)} \sin\frac{\delta}{3} \tag{4.3}$$

and that $|D\bar{u}(\bar{x})| \leq \kappa$ (for $\bar{x} \in \bar{\Omega}_8$) with

$$\kappa = 8 \sin \frac{\delta}{3} + \|Du\|_{L^\infty(B_8)} \cos \frac{\delta}{3}. \tag{4.4}$$

The eigenvalues of the new potential $\bar{u}$ satisfy (2.22) and the interior Hessian bound by Theorem 1.2

$$\left|D^2\bar{u}(\bar{x})\right| \leq C(3) \exp\left[C(3)\kappa^3\right]$$

for $\bar{x} \in \bar{\Omega}_5$. It follows that the induced metric on $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in $\bar{x}$ coordinates is bounded on $\bar{\Omega}_5$ by

$$d\bar{x}^2 \leq \bar{g}(\bar{x}) \leq \mu(\kappa,\delta) d\bar{x}^2, \tag{4.5}$$

where

$$\mu(\kappa,\delta) = \min\left\{1 + C(3)\exp\left[C(3)\kappa^3\right], \left[1 + \left(\cot\frac{\delta}{3}\right)^2\right]\right\}. \tag{4.6}$$

Step 2. Relying on the above set-up and the relative isoperimetric inequality in Proposition 2.3, we proceed with the following Sobolev inequality for functions without compact support.

Proposition 4.1. *Let u be a smooth solution to (1.1) with $\Theta = \pi/2 + \delta$ on $B_5(0) \subset \mathbb{R}^3$. Let f be a smooth positive function on the special Lagrangian surface $\mathfrak{M} = (x, Du(x))$. Then*

$$\left[\int_{B_1} \left|(f-\iota)^+\right|^{3/2} dv_g\right]^{2/3} \leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}\left|\nabla_g(f-\iota)^+\right| dv_g,$$

where ρ, κ, and μ were defined in (4.3), (4.4), and (4.6); also $\iota = \int_{B_5(0)} f dx$.

Proof. Step 2.1. Let $M = ||f||_{L^\infty(B_1)}$. We may assume $\iota < M$. By Sard's theorem, the level set $\{x|\ f(x) = t\}$ is C^1 for almost all t. We first show that for all such $t \in [\iota, M]$,

$$|\{x|\ f(x) > t\} \cap B_1|_g \leq C(3)\rho^6\left[\mu(\kappa,\delta)|\{x|\ f(x) = t\} \cap B_5|_g\right]^{3/2}. \tag{4.7}$$

(Here $|\ |_g$ and $|\ |_{\bar{g}}$ denote the area or volume with respect to the induced metric; $|\ |$ denotes the ones with respect to the Euclidean metric as in Lemma 2.4 and Proposition 2.3 .)

From $t > \int_{B_5} f dx$, it follows that $|\{x|\ f(x) > t\} \cap B_1| < 1$ and consequently

$$|\{x|\ f(x) \leq t\} \cap B_1| > |B_1| - 1 > 1. \tag{4.8}$$

Now we use instead the coordinates for $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ given by the Lewy rotation (4.1). Let

$$A_t = \{\bar{x}|\ f(\bar{x}) > t\} \cap \bar{\Omega}_5,$$

where we are treating f as a function on the special Lagrangian surface $\mathfrak{M}$. Applying Proposition 2.3 with (4.2) and (4.3), we see that

$$\min\left\{\left|A_t \cap \bar{\Omega}_1\right|, \left|A_t^c \cap \bar{\Omega}_1\right|\right\} \leq C(3)\rho^3\left|\partial A_t \cap \partial A_t^c\right|^{3/2}.$$

Now either $\left|A_t\cap\bar{\Omega}_1\right|\leq\left|A_t^c\cap\bar{\Omega}_1\right|$, or vice versa.

If $\left|A_t\cap\bar{\Omega}_1\right|\leq\left|A_t^c\cap\bar{\Omega}_1\right|$, then we have from (4.5)

$$\begin{aligned}\left|A_t\cap\bar{\Omega}_1\right|_{\bar{g}} &\leq[\mu(\kappa,\delta)]^{3/2}\left|A_t\cap\bar{\Omega}_1\right|\\ &\leq C(3)\rho^3\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t\cap\partial A_t^c\right|^{3/2}\\ &\leq C(3)\rho^3\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t\cap\partial A_t^c\right|_{\bar{g}}^{3/2}.\end{aligned}$$

Otherwise, if $\left|A_t\cap\bar{\Omega}_1\right|>\left|A_t^c\cap\bar{\Omega}_1\right|$, still we have that

$$\left|A_t\cap\bar{\Omega}_1\right|\leq C(3)\rho^3\left|A_t^c\cap\bar{\Omega}_1\right|$$

as $\left|A_t^c\cap\bar{\Omega}_1\right|\geq 1/2^3$ from (4.8) and (2.21), and $\rho\geq 8\cos\frac{\pi}{3}$ from (4.3). Thus

$$\begin{aligned}\left|A_t\cap\bar{\Omega}_1\right|_{\bar{g}} &\leq[\mu(\kappa,\delta)]^{3/2}\,C(3)\rho^3\left|A_t^c\cap\bar{\Omega}_1\right|\\ &\leq C(3)\rho^6\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t\cap\partial A_t^c\right|_{\bar{g}}^{3/2}.\end{aligned}$$

In either case we have the desired isoperimetric inequality (now given in the new coordinates for $\mathfrak{M}$) which holds for $\iota<t<M$

$$\left|A_t\cap\bar{\Omega}_1\right|_{\bar{g}}\leq C(3)\rho^6\left[\mu(\kappa,\delta)\left|\partial A_t\cap\partial A_t^c\right|_{\bar{g}}\right]^{3/2},$$

or equivalently (4.7) in the original coordinates.

Step 2.2. With this isoperimetric inequality in hand, the following proof is standard (cf. [LY, Theorem 5.3.1]).

$$\begin{aligned}\left[\int_{B_1}\left|(f-\iota)^+\right|^{3/2}dv_g\right]^{2/3} &=\left(\int_0^{M-\iota}\left|\{x|\ f(x)-\iota>t\}\cap B_1\right|_g dt^{3/2}\right)^{2/3}\\ &\leq\int_0^{M-\iota}\left|\{x|\ f(x)-\iota>t\}\cap B_1\}\right|_g^{2/3}dt\\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_0^{M-\iota}\left|\{x|\ f(x)=t\}\cap B_5\right|_g dt\\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}\left|\nabla_g(f-\iota)^+\right|dv_g,\end{aligned}$$

where the last inequality followed from the the coarea formula; the second inequality from (4.7); and the first inequality from the Hardy-Littlewood-Polya inequality for any nonnegative, nonincreasing integrand $\eta(t)$:

$$\left[\int_0^T\eta(t)^q\,dt^q\right]^{1/q}\leq\int_0^T\eta(t)\,dt.$$

This H-L-P inequality (with $q>1$) is proved by noting that $s\eta(s)\leq\int_0^s\eta(t)\,dt$ and integrating the inequality

$$q\,[s\eta(s)]^{q-1}\,\eta(s)\leq q\left[\int_0^s\eta(t)\,dt\right]^{q-1}\eta(s)=\frac{d}{ds}\left[\int_0^s\eta(t)\,dt\right]^q.$$

The proposition is thus proved. $\square$

Step 3. We continue the proof of Theorem 1.1. As in the proof of Theorem 1.2, we take

$$b=\max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K_\Theta\right\}\quad\text{with}\quad K_\Theta=1+\ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)}.$$

Based on Proposition 2.1, a calculation similar to (3.1) shows that the Lipschitz function $[(b-\iota)^+]^{3/2}$ is weakly subharmonic, where $\iota=\fint_{B_5(0)}bdx$. We apply Michael and Simon's mean value inequality [MS, Theorem 3.4] to obtain

$$\begin{aligned}(b-\iota)^+(0)&\le C(3)\left[\int_{B_1}\left|(b-\iota)^+\right|^{3/2}dv_g\right]^{2/3}\\&\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}\left|\nabla_g(b-\iota)^+\right|dv_g,\end{aligned}$$

where the second inequality follows from Proposition 4.1, approximating $(b-\iota)^+$ by smooth functions if necessary. Thus

$$\begin{aligned}b(0)&\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}|\nabla_g b|\,dv_g+\fint_{B_5}bdx\\&\le C(3)\rho^4\mu(\kappa,\delta)\left(\int_{B_5}|\nabla_g b|^2\,dv_g\right)^{1/2}\left(\int_{B_5}V\,dx\right)^{1/2}+\int_{B_5}V\,dx\\&\le C(3)\rho^4\mu(\kappa,\delta)\int_{B_6}V\,dx\end{aligned}\tag{4.9}$$

where we have used the Jacobi inequality in Proposition 2.1, and a similar calculation leading to (3.2) in the proof of Theorem 1.2.

Step 4. We finish the proof of Theorem 1.1 by bounding $\int_{B_6}V\,dx$. Observe

$$V=\left|\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_3\right)\right|=\frac{\sigma_2-1}{|\cos\Theta|}>0.$$

We control the integral of σ_2 in the following.

$$\begin{aligned}\int_{B_r}\sigma_2dx&=\int_{B_r}\frac{1}{2}\left[(\triangle u)^2-\left|D^2u\right|^2\right]dx\\&=\frac{1}{2}\int_{B_r}div\left[\left(\triangle uI-D^2u\right)Du\right]dx\\&=\frac{1}{2}\int_{\partial B_r}\left\langle\left(\triangle uI-D^2u\right)Du,\nu\right\rangle dA,\end{aligned}$$

where ν is the outward normal of B_r. Diagonalizing D^2u, we see easily that

$$\triangle uI-D^2u=\begin{bmatrix}\lambda_2+\lambda_3&&\\&\lambda_3+\lambda_1&\\&&\lambda_1+\lambda_2\end{bmatrix}>0$$

as $\theta_i+\theta_j>0$ with $\theta_1+\theta_2+\theta_3\geq\pi/2$. Then

$$\int_{B_r}\sigma_2 dx\leq\|Du\|_{L^\infty(\partial B_r)}\int_{\partial B_r}\triangle u dA.$$

Integrating the boundary integral from $r=6$ to $r=7$, we get

$$\begin{aligned}\int_{B_6}\sigma_2 dx &\leq\|Du\|_{L^\infty(B_7)}\min_{r\in[6,7]}\int_{\partial B_r}\triangle u dA\\ &\leq\|Du\|_{L^\infty(B_7)}\int_{B_7}\triangle u dx\\ &\leq C(3)\|Du\|^2_{L^\infty(B_7)}.\end{aligned}$$

It follows that for $\Theta\geq\pi/2$

$$\begin{aligned}\int_{B_6}V dx=\frac{1}{|\cos\Theta|}\int_{B_6}(\sigma_2-1)\,dx &\leq\frac{1}{|\cos\Theta|}\int_{B_6}\sigma_2 dx\\ \leq\frac{C(3)}{|\cos\Theta|}\|Du\|^2_{L^\infty(B_7)}\end{aligned}$$

or

$$\int_{B_6}V dx\leq\frac{C(3)}{|\cos\Theta|\,\|Du\|_{L^\infty(B_8)}}\|Du\|^3_{L^\infty(B_8)}. \tag{4.10}$$

In order to get Θ-independent control on the volume, we estimate the volume in another way. By the Sobolev inequality on the minimal surface $\mathfrak{M}$ [MS, Theorem 2.1] or [A, Theorem 7.3], we have

$$\int_{B_6}V dx=\int_{B_6}dv_g\leq\int_{B_7}\phi^6 dv_g\leq C(3)\left[\int_{B_7}|\nabla_g\phi|^2\,dv_g\right]^3,$$

where the nonnegative cut-off function $\phi\in C_0^\infty(B_7)$ satisfies $\phi=1$ on B_6 and $|D\phi|\leq1.1$.

Observe the (conformality) identity again

$$\left(\frac{1}{1+\lambda_1^2},\cdots,\frac{1}{1+\lambda_3^2}\right)V=$$
$$\left(\sin\Theta(\sigma_1-\lambda_1)+\cos\Theta\left(1-\frac{\sigma_3}{\lambda_1}\right),\cdots,\sin\Theta(\sigma_1-\lambda_3)+\cos\Theta\left(1-\frac{\sigma_3}{\lambda_3}\right)\right),$$

which follows from differentiating the complex identity

$$\ln V+\sqrt{-1}\sum_{i=1}^3\arctan\lambda_i=\ln\left[1-\sigma_2+\sqrt{-1}(\sigma_1-\sigma_3)\right].$$

We then have

$$\begin{aligned}\int_{B_7} |\nabla_g \phi|^2 \, dv_g &= \int_{B_7} \sum_{i=1}^{3} \frac{|\phi_i|^2}{1+\lambda_i^2} V \, dx \\ &\leq 1.21 \int_{B_7} \left[2 \sin \Theta \sigma_1 + \cos \Theta \left(3 - \sigma_2\right)\right] dx \\ &\leq C\left(3\right) \left[\left|\sin \Theta\right| \|Du\|_{L^\infty(B_8)} + \left|\cos \Theta\right| \|Du\|^2_{L^\infty(B_8)}\right]\end{aligned}$$

for $\Theta \geq \pi/2$, where we used the argument leading to (4.10). Thus we get

$$\int_{B_6} V \, dx \leq C\left(3\right) \left[\left|\sin \Theta\right| \|Du\|_{L^\infty(B_8)} + \left|\cos \Theta\right| \|Du\|_{L^\infty(B_8)} \|Du\|_{L^\infty(B_8)}\right]^3 \tag{4.11}$$

Now either $\left|\cos \Theta\right| \|Du\|_{L^\infty(B_8)} \leq 1$ or $\left|\cos \Theta\right| \|Du\|_{L^\infty(B_8)} > 1$; combining (4.10) and (4.11), we have in either case

$$\int_{B_6} V \, dx \leq C\left(3\right) \|Du\|^3_{L^\infty(B_8)}.$$

Finally from the above inequality and (4.9), we conclude

$$\begin{aligned}b(0) &\leq C(3)\rho^4 \mu(\kappa, \delta) \|Du\|^3_{L^\infty(B_8)} \\ &\leq C(3)\rho^4 \|Du\|^3_{L^\infty(B_8)} \min \left\{1 + C(3) \exp\left[C(3)\kappa^3\right], 1 + \left(\cot \frac{\delta}{3}\right)^2\right\}.\end{aligned}$$

Exponentiating, and recalling (4.3), (4.4) and (4.6), we have the Θ-independent bound

$$\left|D^2 u(0)\right| \leq C(3) \exp\left\{C(3) \exp\left[C(3) \|Du\|^3_{L^\infty(B_8)}\right]\right\}$$

and the Θ-dependent bound

$$\left|D^2 u(0)\right| \leq C(3) \exp\left\{C(3) \left[1 + \left(\cot \frac{\delta}{3}\right)^2\right] \left[1 + \|Du\|_{L^\infty(B_8)} \sin \frac{\delta}{3}\right]^4 \|Du\|^3_{L^\infty(B_8)}\right\}.$$

Simplifying the above expressions, we arrive at the conclusion of Theorem 1.1.

5. Proof of Theorem 1.3

We assume that $R = 1$ by scaling $u\left(Rx\right)/R^2$, and $\Theta \geq \left(n-2\right)\pi/2$ by symmetry.

Case $\Theta = \left(n-2\right)\pi/2$. Set $M = \mathrm{osc}_{B_1} u$. We may assume $M > 0$. By replacing u with $u - \min_{B_1} u + M$, we have $M \leq u \leq 2M$ in B_1. Let

$$w = \eta \left|Du\right| + Au^2$$

with $\eta = 1 - |x|^2$ and $A = n/M$. We assume that w attains its maximum at an interior point $x^* \in B_1$, otherwise w would take its maximum on the boundary ∂B_1 and the conclusion would be straightforward. Choose

a coordinate system so that D^2u is diagonalized at x^*. We assume, say $u_n \geq \frac{|Du|}{\sqrt{n}}$ (> 0) at x^*. For all $i = 1, \cdots, n$, we have at x^*

$$0 = w_i = \eta \left|Du\right|_i + \eta_i \left|Du\right| + 2Auu_i,$$

then

$$\frac{u_i u_{ii}}{|Du|} = \left|Du\right|_i = -\frac{\eta_i \left|Du\right| + 2Auu_i}{\eta}. \tag{5.1}$$

In particular, we have $u_{nn} < 0$ by the choice of A. Since the phase $\Theta \geq (n-2)\,\pi/2$, it follows that $\lambda_n = \lambda_{\min}$, $|\lambda_n| \leq \lambda_k$, and

$$g^{nn} = \frac{1}{1+\lambda_n^2} \geq \frac{1}{1+\lambda_k^2} = g^{kk} \tag{5.2}$$

for $k = 1, \cdots, n-1$ at x^*.

Next, we show

$$\triangle_g u \geq 0.$$

When D^2u is diagonalized,

$$\triangle_g u = \sum_{i=1}^n g^{ii} u_{ii} = \sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i^2} = \frac{1}{2}\sum_{i=1}^n \sin(2\theta_i).$$

Let $S \subset \mathbb{R}^n$ be the hypersurface (with boundary) given by

$$S = \left\{ \theta \mid \theta_1 + \theta_2 + \cdots + \theta_n = \frac{\pi}{2}(n-2), |\theta_i| \leq \frac{\pi}{2} \right\},$$

where $\theta = (\theta_1, \cdots, \theta_n)$. Set $\Gamma(\theta) = \frac{1}{2}\sum_{i=1}^n \sin(2\theta_i)$. Suppose that Γ obtains a negative minimum on the interior of S at θ^*. At this point $D\Gamma$ vanishes on $T_{\theta^*}S$, thus we have

$$\cos(2\theta_i) = \cos\left(2\theta_j\right), \text{ then } \theta_i = \pm\theta_j.$$

The only two possible configurations for θ are

$$\theta_1 = \cdots = \theta_n = \frac{(n-2)\,\pi}{2n} \quad \text{or}$$
$$\theta_1 = \cdots = \theta_{n-2} = \frac{\pi}{2},\ \theta_{n-1} = -\theta_n.$$

In either case, Γ is nonnegative. This contradiction allows us to verify the nonnegativity of Γ along the boundary ∂S. It follows easily that $\Gamma \geq 0$ there by induction on dimension n, as

$$\partial S = \bigcup_{k=1}^n \left\{ \theta \mid \theta_1 + \cdots + \widehat{\theta_k} + \cdots + \theta_n = \frac{\pi}{2}(n-3),\ |\theta_i| \leq \frac{\pi}{2} \right\}$$

and $\Gamma\left(\theta_1, \theta_2\right) = 0$ for $\theta_1 + \theta_2 = 0$.

Further, we show

$$\triangle_g \left|Du\right| \geq 0.$$

We calculate

$$\begin{aligned}
\triangle_g |Du| &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}\partial_{\alpha\beta}|Du| = \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta}\left(\frac{u_i u_{i\beta\alpha}}{|Du|}+\frac{u_{i\alpha}u_{i\beta}}{|Du|}-\sum_{j=1}^{n}\frac{u_i u_{i\beta}u_j u_{j\alpha}}{|Du|^3}\right)\\
&= \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta}\left(\frac{u_{i\alpha}u_{i\beta}}{|Du|}-\sum_{j=1}^{n}\frac{u_i u_{i\beta}u_j u_{j\alpha}}{|Du|^3}\right)\\
&\overset{D^2u \text{ is diagonal}}{=} \sum_{\alpha=1}^{n} g^{\alpha\alpha}\frac{\left(|Du|^2-u_\alpha^2\right)\lambda_\alpha^2}{|Du|^3}\geq 0,
\end{aligned}$$

where we used the minimality equation (2.1).

Combining the subharmonicity of u and $|Du|$ with (5.2) and (5.1), we have at x^*

$$\begin{aligned}
0 \geq \triangle_g w &= |Du|\triangle_g \eta + 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha |Du|_\alpha + \underbrace{\eta\triangle_g|Du| + 2Au\triangle_g u}_{\geq 0} + 2A\sum_{\alpha=1}^{n} g^{\alpha\alpha}u_\alpha^2\\
&\geq |Du|\triangle_g\eta + 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha \underline{|Du|_\alpha} + 2A\sum_{\alpha=1}^{n} g^{\alpha\alpha}u_\alpha^2\\
&\geq -2ng^{nn}|Du| - 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha\left(\frac{\eta_\alpha|Du|+2Auu_\alpha}{\eta}\right) + \frac{2A}{n}g^{nn}|Du|^2\\
&\geq -2ng^{nn}|Du| - 4g^{nn}\frac{|Du|}{\eta} - 8g^{nn}Au\frac{|Du|}{\eta} + \frac{2A}{n}g^{nn}|Du|^2;
\end{aligned}$$

It follows that

$$0 \geq -2n\eta - 4 - 8Au + \frac{2A}{n}\eta|Du|.$$

Then by the assumption $M \leq u \leq 2M$ and $A = n/M$

$$\eta|Du|(x^*) \leq (n+2+8n)M.$$

So we obtain

$$|Du(0)| \leq w(x^*) \leq 15nM. \tag{5.3}$$

Case $\Theta > (n-2)\pi/2$. Let $\Theta = \delta + (n-2)\pi/2$. From our special Lagrangian equation (1.1), we know

$$\theta_i + (n-1)\frac{\pi}{2} > (n-2)\frac{\pi}{2} + \delta \quad\text{or}\quad \theta_i > -\frac{\pi}{2} + \delta.$$

We can control the gradient of the convex function $u(x)+\frac{1}{2}\max\{\cot\delta,0\}x^2$ by its oscillation, thus

$$|Du(0)| \leq \operatorname*{osc}_{B_1} + \frac{1}{2}\max\{\cot\delta,0\}. \tag{5.4}$$

In order to get rid of the δ-dependence in the gradient estimate, we need the following.

Proposition 5.1. *Let smooth u satisfy (1.1) with $\Theta-(n-1)\frac{\pi}{2}=\delta\in(0,\pi/4)$ on $B_2(0)$. Suppose that*

$$\operatorname*{osc}_{B_2} u \leq \frac{1}{2\sin\delta}. \tag{5.5}$$

Then

$$|Du(0)| \leq C(n)\left(\operatorname*{osc}_{B_2} u+1\right).$$

Proof. We take the Lewy rotation in the proof of Proposition 2.2, to obtain a "critical" representation $\mathfrak{M}=(\bar{x},D\bar{u}(\bar{x}))$ for the original special Lagrangian graph $\mathfrak{M}=(x,Du(x))$ with $x\in B_2$. Recentering the new coordinates, we take

$$\begin{cases} \bar{x}=x\cos\frac{\delta}{n}+Du(x)\sin\frac{\delta}{n}-Du(0)\sin\frac{\delta}{n} \\ D\bar{u}(\bar{x})=-x\sin\frac{\delta}{n}+Du(x)\cos\frac{\delta}{n} \end{cases}. \tag{5.6}$$

By (2.21) we see that the potential $\bar{u}$ is defined on a ball in $\bar{x}$-space around the origin of radius

$$\bar{R}=\frac{2}{2\cos(\frac{\delta}{n})}>1.$$

From (5.6) and the estimate (5.3) for the critical potential, we have

$$|Du(0)|=\frac{|D\bar{u}(\bar{0})|}{\cos(\delta/n)}\leq C(n)\operatorname*{osc}_{\bar{B}_1}\bar{u}.$$

Next,we estimate the oscillation of $\bar{u}$ in terms of u. We may assume that $\bar{u}(\bar{0})=0$. Without loss of generality we assume the maximum of $|\bar{u}|$ on $\bar{B}_1(\bar{0})$ happens along the positive $\bar{x}_1$-axis, and even on the boundary $\partial\bar{B}_1$. Thus we have

$$\operatorname*{osc}_{\bar{B}_1}\bar{u}\leq 2\left|\int_{\bar{x}_1=0}^{\bar{x}_1=1}\bar{u}_{\bar{x}_1}d\bar{x}_1\right|.$$

In the following, we convert the integral of $\bar{u}_{\bar{x}_1}$ to one in terms of u_{x_1}, then recover the oscillation of $\bar{u}$ from that of u.

We work on the x_1-y_1 plane in the remaining of the proof. Under our above assumption, the $\bar{x}_1$-axis is given by the line

$$y_1=\tan\left(\frac{\delta}{n}\right)x_1$$

and the curve $\gamma:(x_1,u_1(x_1))$ with $|x_1|<2$ forms a graph over the $\bar{x}_1$-axis. Let l_0 be the line perpendicular to the $\bar{x}_1$-axis and intersecting the curve γ at $(0,u_1(0))$ along the y_1-axis. The intersection of l_0 and the $\bar{x}_1$-axis (which is also the origin of the recentered the $\bar{x}_1$-$\bar{y}_1$ plane) has distance to the origin of the x_1-y_1 plane given by

$$|u_1(0)|\sin\left(\frac{\delta}{n}\right)\leq\left(\operatorname*{osc}_{B_1}u+\frac{1}{2}\cot\delta\right)\sin\left(\frac{\delta}{n}\right)\leq 1 \tag{5.7}$$

by the rough bound (5.4) and the condition (5.5). Now let l_1 be the line parallel to l_0 passing through the point $\bar{x}_1=1$ along the $\bar{x}_1$-axis.

The integral

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} d\bar{x}_1$$

is the signed area between the $\bar{x}_1$-axis and the curve γ, and lying between the lines l_0 and l_1. We convert this to an integral over x_1,

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} d\bar{x}_1 = \int_{P(l_0\cap\bar{x}_1\text{-axis})}^{P(l_1\cap\bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right)x_1\right] dx_1 + K_0 + K_1,$$

where P denotes projection to the x_1-axis, and K_0 as well as K_1 denotes the signed areas to the left or right of the desired region, forming the difference.

It is important to note the following for $j = 1, 2$:

(i) $P(l_j \cap \bar{x}_1\text{-axis})$ is in the x_1-domain of u_1 by (5.7),

$$|P(l_0 \cap \bar{x}_1\text{-axis})| \le 1 \cdot \cos\left(\frac{\delta}{n}\right) < 1,$$
$$|P(l_1 \cap \bar{x}_1\text{-axis})| \le (1+1) \cdot \cos\left(\frac{\delta}{n}\right) < 2;$$

(ii) $P(l_j \cap \gamma)$ is also in the x_1-domain of u_1 as the whole Lagrangian surface $\mathfrak{M}$ is a graph over B_2,

$$|P(l_j \cap \gamma)| \le 2;$$

(iii) the region K_j is bounded by the line l_j, the vertical line $x_1 = P(l_j \cap \bar{x}_1$-axis), and the curve γ, also each region K_j is on one side of the $\bar{x}_1$-axis.

Thus from (i)

$$\left|\int_{P(l_0\cap\bar{x}_1\text{-axis})}^{P(l_1\cap\bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right)x_1\right] dx_1\right| \le \operatorname*{osc}_{B_2} u + C(n)$$

and from (ii) (iii)

$$|K_j| \le \left|\int_{P(l_j\cap\bar{x}_1\text{-axis})}^{P[l_j\cap\gamma]} \left[u_1(x_1) - \tan(\frac{\delta}{n})x_1\right] dx_1\right| \le \operatorname*{osc}_{B_2} u + C(n).$$

It follows that we have the conclusion of Proposition 5.1

$$|Du(0)| \le C\,(n) \operatorname*{osc}_{\bar{B}_1} \bar{u} \le C\,(n)\left(\operatorname*{osc}_{B_2} u + 1\right).$$

□

We finish the proof of Theorem 1.3. For $\delta \ge \pi/4$, the bound (5.4) gives

$$|Du\,(0)| \le \operatorname*{osc}_{B_1} u + \frac{1}{2} \le C(n)\left[\operatorname*{osc}_{B_2} u + 1\right].$$

For $\delta \le \pi/4$, if $\operatorname{osc}_{B_2} u \le 1/\left(2\sin\delta\right)$, then Proposition 4.1 gives

$$|Du(0)| \le C(n)\left[\operatorname*{osc}_{B_2} u + 1\right].$$

Otherwise, $\operatorname{osc}_{B_2} u > 1/(2\sin\delta)$, and from (5.4)

$$|Du(0)| \le \underset{B_1}{\operatorname{osc}}\, u + \underset{B_2}{\operatorname{osc}}\, u \le C(n)\left[\underset{B_2}{\operatorname{osc}}\, u + 1\right].$$

Applying this estimate on $B_2(x)$ for any $x \in B_1(0)$, we arrive at the conclusion of Theorem 1.3.

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